AN ARITHMETICAL APPROACH TO ORDINARY FOURIER SERIES

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BY

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FOREWORD

The formal background of all the following considerations suggests itself, and is substantially contained in, what the astronomers and the geophysicists call harmonic analysis. It is something quite different from what is meant by harmonic analysis in physical wave theories (for instance, the Fourier decompositions of electromagnetic or acoustical vibrations), in which the issues, being mainly *theoretical* in nature, can hardly be distinguished from what is dealt with in the Fourier theories of the pure mathematicians. In contrast, an analysis of periodicities involving planetary motions or tidal effects must take into account such a huge *empirical* material, and require in turn such a high degree of *numerical* precision of the result, that the astronomers have been compelled to devise their own methods, appropriate for dealing with situations of this type. Cf. the introductory presentation in [1].

This development took place long before astronomer and astrophysicist became synonymous terms. The astronomers had to apply their own mathematical ingenuity, but not because they were unfamiliar with what they could have learned from mathematical papers (as a matter of fact, the astronomical work of Lagrange and of Euler had a greater influence on what today is called Fourier theory than is usually realized). Correspondingly, the formal procedures underlying the numerical schemes in question make good mathematical sense. Nevertheless, the subsequent mathematical literature appears completely to ignore these developments.

The reasons for this (as will be seen, quite undeserved) disregard may be various. For example, if a mathematician is curious enough to have a glance at one of the investigations in question, he will not be attracted by the length or by the deduction of the formulae, the substance of which he could indeed obtain in a more elegant fashion (though he is likely to forget that his concise formulae may call for arrangements which can in no way be adjusted to the needs of actual computation; a natural instance of such a situation will arise below). He may also dislike the complete lack of convergence proofs. And he may become so disgusted as to close the book, if he arrives at a passage (such as the one just at the beginning of the crucial §95 in [1]) which, in an obvious effort to appease precisely him, volunteers a sufficient criterion which he either knows to be insufficient or is ready to disprove on demand.

Inasmuch as all of this misses the whole point, the subject seemed to be worth presenting in a form acceptable to a mathematical conscience. This aim was the starting point of the following considerations.

It turned out that, while part of the facts needed are elementary in nature (elementary in the technical sense of the analytic theory of numbers), another part of the issues involved leads to questions depending on the Prime Number Theorem, and even on certain refinements of it.

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1. Let $f(x)$, $-\infty < x < \infty$, be a function of period 1. Then the same is true of each of the functions

$$(1) \qquad f_n(x) = \sum_{m=1}^{n} f(x + m/n)/n.$$

If $f(x)$ is R-integrable (over a period), then the function (1) tends to the constant

$$(2) \qquad \int_x^{x+1} f(t)\, dt = \int_0^1 f(t)\, dt$$

as $n \to \infty$. The rapidity of the convergence of the equidistant (R)-sum $f_n(0)$ to the integral is easily realized to involve

(α) the smoothness properties of $f(x)$ for $0 \leqq x < 1$ (such as finiteness of the total variation of $f(x)$ or the existence of a uniformly continuous first derivative; cf. (i) below) and

(β) the periodicity condition $f(0) = f(1)$.

That an increase in the smoothness (e.g., the requirement of a uniformly continuous derivative of higher order) is, in itself, incapable of improving the rapidity of the convergence of $f_n(0)$ to (2), is implied by asymptotic relations of Pólya [10], which assume only (α).

It turns out that this situation completely changes if (β) is required at the same time. In fact, the violation of (β) effects something like a Gibbs phenomenon, as $n \to \infty$ in $f_n(0)$. Actually, the problem is not only paralleled by, but can also be reduced to, questions pertaining to the rapidity of the convergence of the Fourier series of $f(x)$; questions which involve both (α) and (β). The formal explanation of this connection will be exhibited by (iii). However, for reasons which will become apparent below, the Fourier approach to the problem will be avoided for the present.

For the present, all that will be needed is a sufficiently strong condition (α) by virtue of which the deviations of $f_1(x), f_2(x), \cdots$ from (2) form an absolutely-uniformly convergent series. In view of the Gibbs phenomenon referred to before, such a criterion must necessarily assume the additional restriction (β). The simplest criterion of the desired type is the following:

(i) *If a function $f(x)$ of period 1 has a derivative satisfying a uniform Lipschitz condition, then the error of the approximation (1) to (2) is $O(1/n^2)$, uniformly in x, as $n \to \infty$.*

More explicitly, if $f(x)$ is real-valued (which involves no loss of generality) and if C denotes the Lipschitz constant of the derivative $f'(x)$, then $f_n(x)$ cannot differ from (2) by more than C/n^2.

2. In order to prove this, it is sufficient to show that the assumption of periodicity and the existence of a constant C, satisfying the inequality

$$(3) \qquad |f'(x_1) - f'(x_2)| \leqq C\,|x_1 - x_2|$$

for any pair of values x_1, x_2, imply that $|f_n(0)| \leqq C/n^2$ holds for every n. In fact, $|f_n(t)| \leqq C/n^2$ then follows for every t if $f(x)$ is replaced by $f(x + t)$.

In addition, it can be assumed that the mean-value (2) vanishes. In fact, if $f(x)$ is a constant, then (1) is the same constant, and it can be subtracted from $f(x)$, since the functional (1) of $f(t)$ is distributive.

For a fixed n, let $f'(\alpha_m)$ denote the minimum, and $f'(\beta_m)$ the maximum, of $f'(x)$ on the interval contained between $x = (m - 1)/n$ and $x = m/n$, where $m = 1, 2, \cdots, n$. Then

$$f(m/n)/n - \int_{(m-1)/n}^{m/n} f(x)\,dx$$

$$= \int_{(m-1)/n}^{m/n} \{f(m/n) - f(x)\}\,dx = \int_{(m-1)/n}^{m/n} (m/n - x)f'(\theta_m)\,dx,$$

where $\theta_m = \theta_m(x)$ is on the m-th subinterval. Since the value of the last integral is not less than

$$f'(\beta_m) \int_{(m-1)/n}^{m/n} (m/n - x)\,dx,$$

which in turn is identical with $f'(\beta_m) \int_0^{1/n} x\,dx = \frac{1}{2}f'(\beta_m)/n^2$, the inequality

$$f(m/n)/n - \int_{(m-1)/n}^{m/n} f(x)\,dx \leqq \tfrac{1}{2}f'(\beta_m)/n^2$$

holds for every m. It follows therefore from the case $x = 0$ of the definition (1), and from the vanishing of the integral (2), that

$$f_n(0) \leqq \tfrac{1}{2}\sum_{m=1}^{n} f'(\beta_m)/n^2.$$

Clearly, this inequality remains true if the sign $\leqq$ and the maxima $f'(\beta_m)$ are replaced by $\geqq$ and $f'(\alpha_m)$ respectively. On the other hand, (3) implies that $f'(\beta_m)$ and $f'(\alpha_m)$ deviate from $f'(m/n)$ by not more than C/n. This supplies for $f_n(0)$ the upper and lower bounds

$$\tfrac{1}{2}\sum_{m=1}^{n} (f'(m/n) \pm C/n)/n^2 = \tfrac{1}{2}\sum_{m=1}^{n} f'(m/n)/n^2 \pm \tfrac{1}{2}C/n^2.$$

Consequently, the proof of the inequality $|f_n(0)| \leqq C/n^2$ will be complete if it is verified that the sum multiplying the first $\frac{1}{2}/n^2$ on the right of the last formula line does not exceed C in absolute value. And this will be verified if it is shown that

$$\left|\sum_{m=1}^{n} f'(m/n) - \frac{1}{n}\int_0^1 f'(x)\,dx\right| \leqq C.$$

In fact, the integral inserted vanishes in virtue of the periodicity assumption, $f(0) = f(1)$. But (3) obviously implies each of the m inequalities

$$\left| f'(m/n) - \frac{1}{n} \int_{(m-1)/n}^{m/n} f'(x)\, dx \right| \leqq C/n,$$

and so the assertion follows by summation with respect to m.

This completes the proof of (i).

3. A *legitimate* application of Möbius' inversion leads from (i) to a connection between number theory and ordinary Fourier series. The substance of this connection may be formulated as follows:

(ii) *Let $f(x)$ be a periodic function, of period 1, which possesses a derivative satisfying a uniform Lipschitz condition, and let $f(x)$ be normalized by $a_0 = 0$, where a_0 denotes the mean-value*

$$a_0 = \int_x^{x+1} f(t)\, dt = \int_0^1 f(t)\, dt \tag{4}$$

(i.e., let the constant (4) *be subtracted from the given function f). Let $f_n(x)$, where $n = 1, 2, \cdots$, denote Riemann's n-th equidistant approximation,* (1), *to the first of the integrals* (4), *and let $\mu(n)$ be Möbius' factor. Then*

(I) *each of the series*

$$g_n(x) = \sum_{k=1}^{\infty} \mu(k) f_{nk}(x) \tag{5}$$

is absolutely-uniformly convergent and the function admits, in terms of the functions (5), *the expansion*

$$f(x) = \sum_{n=1}^{\infty} g_n(x), \tag{6}$$

which is absolutely-uniformly convergent; in addition,

(II) *if $f(x)$ is a trigonometric polynomial or, more generally, any function admitting an appropriately rapid uniform approximation by trigonometric polynomials of increasing degree (and, as a matter of fact, the Lipschitz condition imposed on $f'(x)$ turns out to be sufficient for such an approximability), then the n-th of the functions* (5) *satisfies the differential equation*

$$g_n'' + 4\pi^2 n^2 g_n = 0 \tag{7}$$

i.e., the given function $f(x)$ determines two sequences of integration constants a_1, a_2, $\cdots$, b_1, b_2, $\cdots$ by means of which the terms of the expansion (6) *are representable in the form*

$$g_n(x) = a_n \cos 2\pi n x + b_n \sin 2\pi n x. \tag{7 bis}$$

The trouble is that the parenthetical remark of (II) does not seem to be capable of a *direct* proof. The other assertions of (II), and (I) itself, will be

verified in an arithmetical fashion and *without* any reference to the fact that every sufficiently smooth periodic function can be expanded into a Fourier series. If the differential equations (7) could be verified for the functions (5) when all that is known of the given periodic function $f(x)$ occurring (via the definition (1) of $f_1, f_2, \cdots$) in (5) is that $f(x)$ is sufficiently smooth (say of class C'' or, for that matter, of class $C^{(8)}$), then the solution (7 bis) of (7), when substituted into (6), would *prove* that any such $f(x)$ can be expanded into a Fourier series. Actually, the parenthetical remark of (II) is known to imply Weierstrass' approximation theorem, and to be implied by most of the proofs of this theorem. Accordingly, the substantial content of the parenthetical remark of (II) is the fact that the space of the periodic continuous functions has an enumerable basis. And it is just this fact of separability (a fact which, incidentally, implies the separability of the (L^p)-spaces also) that will escape the direct or arithmetical proof.

4. Let $d(k)$, where $k = 1, 2, \cdots$, denote the number of the divisors of k. Then, as is well known (and quite obvious), the estimate

$$d(k) = O(k^\epsilon) \tag{8}$$

holds for every fixed $\epsilon > 0$. Since the assumptions of (ii) imply, by (i), that

$$f_k(x) = O(k^{-2}) \text{ uniformly in } x, \tag{9}$$

it follows that

$$\sum_{k=1}^{\infty} d(k) \mid f_k(x) \mid < \infty, \tag{10}$$

and that the convergence of the series (10) is uniform (in x). This proves the first assertion of (I), since every factor $d(k)$ in (10) is at least 1, while every factor $\mu(k)$ occurring in (5) is either 0 or ± 1.

It also follows that the series on the right of (6) is absolutely-uniformly convergent. In fact, (5) shows that the series (6) is majorized by the double series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mid \mu(k) f_{nk}(x) \mid . \tag{11}$$

But, if $\mu(k)$ in (11) is replaced by its majorant 1, it is seen from the definition of $d(k)$ that the double series (11) is majorized by the simple series (10).

Finally, the definition (5) and the convergence of the non-negative series (11) assure the legitimacy of the rearrangement

$$\sum_{n=1}^{\infty} g_n(x) = \sum_{m=1}^{\infty} f_m(x) \sum_{d \mid m} \mu(d), \tag{12}$$

where the index, d, of the interior summation on the right runs through all divisors of m. Since, by the definition of Möbius' factor,

$$\sum_{d \mid m} \mu(d) = \begin{cases} 1 \text{ if } m = 1, \\ 0 \text{ if } m > 1, \end{cases} \tag{13}$$

and since $f_1(x)$ in (1) denotes the function $f(x)$, the identity (6) follows from (12).

This proves (I). In order to prove (II) for the case of an ordinary trigonometric polynomial $f(x)$ of period 1, it is sufficient to prove it for the case $f(x) = e(mx)$, where $e(t)$ is an abbreviation for $e^{2\pi it}$ and m denotes any fixed positive integer. The possibility of this reduction is clear from the fact that all the operations (1), (5), (7) to which $f(x)$ is subjected are distributive (and such as to be real when $f(x)$ is real; for the latter reason, $e(-mx)$ can be reduced to $e(mx)$, where $m = 0$ is excluded by the assumed vanishing of the mean-value a_0). Accordingly, the assertion to be proved is that, if $f(x) = e(mx)$, where m is any *fixed* positive integer, then (7) is an identity in x for *every* positive integer n. But this can be verified as follows:

If $f(x) = e(mx)$, then, according to (1),

$$f_k(x) = \sum_{j=1}^{k} e(m(x + j/k))/k = e(mx) \sum_{j=1}^{k} e(mj/k)/k.$$

Hence, if k is replaced by nk, it follows that every term of each of the series (5), where $n = 1, 2, \cdots$, and therefore each of the series $g_n(x)$, which are convergent by (I), is a constant multiple of $e(mx)$. However, although the second derivative of $e(x)$ is $-4\pi^2 e(x)$, this representation of $g_n(x)$ does not prove (7) for every n, but merely for $n = m$. Correspondingly, what remains to be verified is the identical vanishing of $g_n(x)$ for every $n \neq m$. Actually, the following verification will be such as to apply to the exceptional case $n = m$ also.

If k is replaced by nk in the last formula line, it is seen that $f_{nk}(x)$ is $e(mx)$ times the mean-value of the m-th powers of all nk-th roots of unity. This means that $f_{nk}(x)$ is $e(mx)$ or 0 according as the fixed positive integer m is or is not a multiple of the product nk. It follows therefore from (5) that $g_n(x)$ is $e(mx)$ times the constant

$$\sum_{kn \mid m} \mu(k),$$

where the summation index k runs through those positive integers for which the product kn becomes a divisor of m.

In particular, this constant is 0 whenever the quotient m/n is not an integer. In the remaining case, where m/n is an integer, the replacement of m by m/n in (13) shows that the constant represented by the last formula line is still 0 if the case $m/n = 1$, corresponding to the first line on the right of (13), is excepted; in which case (that is, if $m = n$) the sum becomes $\mu(1) = 1$. Accordingly, $g_n(x)$ is $e_m(x)$ times the constant 1 or the constant 0 according as the two positive integers n, m are or are not equal. Hence, $g_n(x)$ satisfies the differential equation (7) of an n-th harmonic vibration, (7 bis), whether n be equal to or distinct from the given m.

5. In view of the methodical comments made after (ii), it is instructive to compare the approach of (ii) to Fourier series with two formal considerations which are due to Fourier and to Bruns respectively.

Just as the approach of (ii), the approach of §215–§220 in Fourier's *Théorie* [5] attempts to *deduce* the existence of a trigonometric expansion (for a particular, but basic, periodic function f). In fact, without assuming for $f(x)$ a Fourier series, Fourier arrives at a formal expansion (6) for the terms of which he verifies the differential equations (7), which are equivalent to (7 bis). However, in contrast to the "arithmetical" definition (5), (1) of the above expansion (6), the terms $g_n(x)$ of Fourier's formal identity for $f(x)$ are obtained in a purely "analytical" manner, as follows: Fourier assumes that $f(x)$ is given as a power series (and, in view of the possibility of an easy modernization by summation methods of divergent series, it is today quite immaterial that this assumption happens to be false precisely for his f). And he obtains the terms (7 bis) of (6) by analytical expansions and subsequent rearrangements of all the terms of this power series. Correspondingly, while an application of (7 bis), (5) and (1) at $x = 0$ simply gives

$$na_n = \sum_{k=1}^{\infty} \mu(k)/k \sum_{m=1}^{kn} f(m/kn), \qquad (n = 1, 2, \cdots), \tag{14}$$

Fourier's corresponding representation of the n-th coefficient is a (divergent) series the terms of which contain the derivatives, of arbitrarily high order, of $f(x)$ at both end-points of the x-interval of periodicity. In other words, while (14) contains the values of $f(x)$ only, *Fourier's analogue of* (14) *is what would be supplied directly by the infinite series of the Euler-Maclaurin sum-formula*, when the latter is applied to the integral representation of the Fourier constants.

In contrast to this approach of Fourier and to (ii), Bruns [1] *assumes* that the function $f(x)$ is expanded into a Fourier series. All that he wants (§93–§98) is a method of calculating the Fourier constants in cases in which the integral representation of the sequence of these constants becomes intractable for the computer. Such a situation arises when the variable $f(x) = f(x + 1)$ is given (measured or tabulated) for a *large* number of equidistant x-values, say for $x = h/N$, where $h = 1, \cdots, N$, and the numerical harmonic analysis of $f(x)$ therefore requires the calculation of all coefficients occurring in a *high* partial sum of the Fourier series of $f(x)$. For the case of rational, rather than of trigonometric, approximations, this practical problem was considered already by Tchebyshev [14]; the numerical gains of his scheme were tried out, before Bruns, by the astronomers Radau [11] and Harzer [7].

Since Bruns assumes that $f(x)$ has a Fourier series, he does not of course consider anything like (7). But he considers the series (5) for *numerical* values of x (such as $x = 0, \frac{1}{2}$). And he deduces (14) as a representation of the Fourier cosine coefficients which takes into account the numerical needs in case of situations of the type just described (even though he does not call the factors involved, which *are* identical with the factors $\mu(m)$ in (14), "Möbius factors", and even though he mentions for the legitimacy of the evaluation (14) of a_n a condition which (v) below will prove to be insufficient). In addition, he develops rules corresponding to (14) but evaluating b_n, the sine coefficients.

Since (14) results by choosing $x = 0$, while every x was allowed above, Bruns'

complicated rules for the calculation of b_n seem to be awkward. One could say that, simply because the sum of the series (5) *is* the harmonic vibration (7 bis), *both* coefficients a_n, b_n are determined if the terms (1) occurring in (5) are available. However, this elegant conclusion does not benefit the computer. In fact, while the single choice $x = 0$ in (7) supplies every a_n, there exists no fixed x (or fixed finite set of x-values) supplying b_n for every n.

6. From now on it will be *granted* that the periodic function $f(x)$ can be expanded into a Fourier series. Then the smoothness assumptions of (ii) can be greatly relaxed. This will be made possible by the following formal connection between the expansion of $f(x)$ and the sequence of the approximative mean-values (1):

(iii) *If a trigonometric series*

$$\sum_{k=-\infty}^{\infty} c_k e(kx), \tag{15}$$

where $\sum_{k=-\infty}^{\infty} = \lim_{m\to\infty} \sum_{k=-m}^{m}$,

$$c_0 = 0 \tag{16}$$

and $e(t) = e^{2\pi it}$, *is convergent for every x, and if $f(x)$ denotes its sum, then the trigonometric series*

$$\sum_{k=-\infty}^{\infty} c_{nk} e(nkx) \tag{17}$$

converges for every n and for every x, and its sum is $f_n(x)$. If (15) *is the Fourier series* (*of class* (*L*) *or of any "additive" class*) *of a function $f(x)$, then* (17) *belongs as Fourier series* (*of the same class*) *to the function $f_n(x)$. In either case, $f_n(x)$ denotes the function* (1). (In the first case, (15) need not be a Fourier series; in the second case, (15) need not converge.)

If (iii) is applied at $x = 0$ in the first case, (1) shows that the assertion of (iii) becomes the identity

$$\frac{1}{n}\sum_{m=1}^{n} f(m/n) = \sum_{k=-\infty}^{\infty} c_{nk} \tag{18}$$

in n. And (18) is nothing but Poisson's (or Dirichlet's; cf. [3]) sum-formula, of which the Euler-Maclaurin sum-formula is a particular case (applying when $f(x)$ has the derivatives required by the partial integrations involved; cf., e.g., [8]). Actually, the proof of (iii) will be the same as in the case $x = 0$ of (18), except that it will be shorter in one respect. In fact, the possibility of a simplification of the classical proof (cf. [8]) is disguised precisely by the restriction of the independent variable x to the particular point $x = 0$.

Another consequence of (iii) seems to be worth formulating as a

COROLLARY. *In order that*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series of a function of bounded variation, it is necessary that, besides the coefficient estimates

$$a_n = O(1/n), \qquad b_n = O(1/n),$$

the estimates

$$C_n(x) = O(1/n), \qquad S_n(x) = O(1/n)$$

hold, uniformly in x, for the series

$$C_n(x) = \sum_{k=1}^{\infty} a_{nk} \cos kx, \qquad S_n(x) = \sum_{k=1}^{\infty} b_{nk} \sin kx,$$

which must converge for every n and for every x. (For instance, the series

$$\sum_{k=1}^{\infty} a_{nk}, \qquad \sum_{k=1}^{\infty} (-1)^k b_{n(2k-1)}$$

must converge for every n, and their sums must be $O(1/n)$ as $n \to \infty$.)

7. In order to prove (iii), suppose that (15) converges for every x. Then, if $f(x)$ denotes the sum of (15), the replacement of x by $x + m/n$ gives

$$f(x + m/n) = \sum_{k=-\infty}^{\infty} c_k e(kx) e(km/n).$$

Hence, from (1),

$$f_n(x) = \sum_{j=-\infty}^{\infty} c_j e(jx) \sum_{m=1}^{n} e(jm/n)/n.$$

But the interior sum on the right, being the mean-value of the j-th powers of all n-th roots of unity, is 1 or 0 according as j is or is not of the form nk, where $k = 0, \pm 1, \cdots$. It follows therefore from (16) that the expression on the right of the last formula line is identical with the series (17). This proves (iii) for its first case. Its second case follows in the same way.

In order to verify the Corollary, suppose that a function $f(x)$, of period 1, is of bounded variation. Then, according to a well-known observation of Pólya [10],

$$\left| \sum_{m=1}^{n} f(m/n) - n \int_0^1 f(t)\, dt \right| \leqq \text{const.}$$

as $n \to \infty$ (simply because the expression on the left is majorized by

$$n \sum_{m=1}^{n} \int_{(m-1)/n}^{m/n} | f(m/n) - f(t) |\, dt \leqq n \sum_{m=1}^{n} \int_{(m-1)/n}^{m/n} | df(t) |/n = \text{const.}$$

the const. being the total variation of $f(t)$ over a period). Hence, if $f(t)$ is replaced by $f(t + x)$, it is seen from (1) that $| nf_n(x) | \leqq$ const. holds for every x and for every n, provided that it is assumed that the integral (2) vanishes, i.e., that (16) is satisfied. But (iii) then ensures that $f_n(x)$ is identical with (17). Hence, if (16) is satisfied, the sum of the series (17), and therefore the sum of the series

$$\sum_{k=-\infty}^{\infty} c_{nk}\, e(kx)$$

as well, is majorized by const./n. This proves the Corollary, the assertions of which are indeed worded in such a manner that the additive term $\frac{1}{2}a_0$, occurring in the assumption (16), does not appear at all.

It follows from the Corollary that *there exist continuous, periodic functions which, without being of bounded variation, possess absolutely convergent Fourier series having coefficients which are positive and of the order* $o(1/n)$. In fact, it is easy to construct a numerical sequence $a_1, a_2, \cdots$ showing that the three conditions $a_n > 0$, $na_n \to 0$, $a_1 + a_2 + a_3 + \cdots < \infty$ are compatible with $\limsup n(a_n + a_{2n} + a_{3n} + \cdots) = \infty$. But the Corollary implies that $1 + \sum (a_n \cos nx + 2^{-n} \sin nx)$ then is a Fourier series of the desired type.

8. Applications of (iii) in quite another direction result *if the sequence of the coefficients* c_n *of* (15) *is subject to arithmetical restrictions.* In this direction, two consequences of (iii), both of which claim *a simple functional equation for the corresponding function* (15), are of particular interest. The second of these consequences of the formal fact expressed by (iii) will contain an extension of the functional equation

$$B_j(nx) = n^{j-1} \sum_{m=0}^{n-1} B_j(x + m/n) \tag{19}$$

of the Bernoullian polynomials $B_1(x)$, $B_2(x)$, $\cdots$ and will exhibit the general arithmetical background of (19). The first may be formulated as follows:

(I) *Let two sequences of constants* $\alpha_2, \alpha_3, \alpha_5, \cdots, \beta_2, \beta_3, \beta_5, \cdots$ *be such as to make convergent (almost everywhere) the trigonometric series*

$$\sum_p (\alpha_p \cos 2\pi px + \beta_p \sin 2\pi px),$$

in which p *runs through the (monotone) sequence of all primes. Then, if* $f(x)$ *denotes the sum of the series, and if* n *is any integer greater than* 1, *the function*

$$\sum_{m=1}^{n} f(x + m/n)$$

vanishes identically (for almost all x*).*

In order to conclude this from (iii), it is sufficient to observe that the present assumption consists in the vanishing of all those (though not necessarily only those) coefficients c_k of (15) for which k is not a prime; an assumption equivalent to the vanishing of all coefficients c_{nk} of all series (17) belonging to any $n \neq 1$.

Actually, (I) can be thought of as just a limiting case of what results by a "logarithmization" of the following fact:

(II) *Let a pair of constants* α, β *and a sequence of constants* $\lambda_1, \cdots, \lambda_k, \cdots$ *be such as to make convergent (almost everywhere) the trigonometric series*

$$\sum_{k=1}^{\infty} (\alpha\lambda_k \cos 2\pi kx + \beta\lambda_k \sin 2\pi kx),$$

and let λ_k *be a completely multiplicative function of* k. *Then, if* $f(x)$ *denotes the sum of the series, and if* n *is any positive integer,*

$$\sum_{m=1}^{n} f(x + m/n) = n\lambda_n f(nx)$$

is an identity (for almost all x*).*

It is understood that by a completely multiplicative function λ_k of k is meant any multiplicative representation of the ordinary multiplication on the set of all positive integers, that is, any sequence $\lambda_1, \lambda_2, \cdots$ satisfying $\lambda_{jk} = \lambda_j\lambda_k$ for all pairs of positive integers j, k (this implies that λ_1 must be 1 if the trivial case $\lambda_1 = 0$, necessitating the vanishing of every λ_k, is excluded).

The simplest example of a multiplicative function λ_k is any fixed power of k. But if $\lambda_k = k^{-j}$, the trigonometric series occurring in (II) becomes exactly the Fourier series of the j-th Bernoullian polynomial $B_j(x)$ for $0 < x < 1$, with $\alpha = 0$ or $\beta = 0$ according as j is odd or even. Hence, (19) is the simplest particular case of (II). In addition, the Bernoullian restriction of α, β by the parity of j becomes superfluous. It is not even necessary to restrict j in $\lambda_k = k^{-j}$ to positive integers, since, $1^{-\epsilon}, 2^{-\epsilon}, \cdots$ being a monotone zero-sequence for every $\epsilon > 0$, both series

$$\sum_{k=1}^{\infty} k^{-j} \cos 2\pi kx, \qquad \sum_{k=1}^{\infty} k^{-j} \sin 2\pi kx$$

converge for $0 < x < 1$ whether the positive number j be an integer or not. Thus (II) extends (19) from the case of the Bernoullian polynomials to the case of the Bernoullian transcendents which occur in Fourier theory of fractional differentiation. Actually, the functional equation of (II) is not restricted to these generalized Bernoullian functions, since it applies to such cases as $\lambda_n = \lambda(n)/n$, where $\lambda(n)$ is Liouville's arithmetical function.

In fact, the most general coefficient sequence satisfying the arithmetical assumption of (II) results by assigning the value of λ_p (in an arbitrary manner) for every prime p, and then placing $\lambda_n = \lambda_p^k$ if $n = p^k$, where $k = 1, 2, \cdots$, and $\lambda_n = \lambda_r\lambda_s \cdots$ if $n = rs \cdots$, where $r = p^k$, $s = q^j, \cdots$ denote powers of distinct primes $p, q, \cdots$. The remaining assumption of (II), that concerning convergence almost everywhere, is certainly satisfied if the arbitrary data λ_p are so chosen as to make the series

$$\sum_{n=1}^{\infty} |\lambda_n|^2 \log n$$

convergent. (This condition is satisfied if $\lambda_n = \lambda(n)/n$, and even if $\lambda_n = \lambda(n)/n^{\frac{1}{2}+\epsilon}$, but it is not satisfied in the Bernoullian case $\lambda_n = n^{-\epsilon}$, if $\epsilon \leqq \frac{1}{2}$.) In terms of the independent data λ_p, a sufficient condition is represented by the convergence of the series

$$\sum_{p} |\lambda_p|^2 p^{\epsilon}$$

for some positive ϵ. This follows by estimating the factor $\log n$ of the terms of the preceding series by $O(n^\epsilon)$ and then observing that, since λ_n and n^ϵ are completely multiplicative functions of n,

$$\sum_{n=1}^{\infty} |\lambda_n|^2 n^\epsilon = \prod_p (1 + |\lambda_p|^2 p^\epsilon + |\lambda_p|^4 p^{2\epsilon} + \cdots)$$

(Euler).

The proof of (II) itself results by writing out (17) in the present notations,

$$\sum_{k=1}^{\infty} (\alpha\lambda_{nk} \cos 2\pi nkx + \beta\lambda_{nk} \sin 2\pi nkx).$$

Since $\lambda_{nk} = \lambda_n\lambda_k$, it is seen that (17) becomes identical with λ_n times the series which is obtained when x is replaced by nx in the series defining $f(x)$. It follows therefore from the definition (1), that (II) is a corollary of (iii).

9. Only (iii) itself, rather than any of its above consequences, will be needed in the proof of the following extension and precision of the assertions of (ii):

(iv) *Let a continuous function $f(x)$, of period 1 and of vanishing mean-value (4), possess a Fourier series*

$$f(x) = \sum_{n=1}^{\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx) \tag{21}$$

in which the Fourier constants satisfy the condition

$$\sum_{n=1}^{\infty} 2^{\nu(n)} (|a_n| + |b_n|) < \infty, \tag{22}$$

where $\nu(n)$ denotes the number of the distinct prime divisors of n. Then the n-th term of (21) can be represented in the form

$$a_n \cos 2\pi nx + b_n \sin 2\pi nx = \sum_{k=1}^{\infty} \mu(k) f_{nk}(x) \tag{23}$$

for every x, where $\mu(k)$ is Möbius' factor, the functions $f_n(x)$ denote the equidistant Riemannian approximations (1) to the mean-value (4), and the convergence of the series (23) is part of the statement.

A sufficient condition for (23) consists in the existence of an $\epsilon > 0$ satisfying

$$\sum_{n=1}^{\infty} n^\epsilon (|a_n| + |b_n|) < \infty \tag{24}$$

(and so, in particular, in the assumption that the function $f(x)$ satisfies a fractional Lipschitz condition

$$|f(x_1) - f(x_2)| = O(|x_1 - x_2|^{\frac{1}{2}+\epsilon}), \tag{25}$$

uniformly in x_1, x_2 as $|x_1 - x_2| \to 0$, where $\epsilon > 0$).

On the other hand, the case $\epsilon = 0$ *of* (24), *i.e., the assumption*

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty, \tag{26}$$

(*or, what is the same thing, the absolute convergence of the Fourier series*) *is an insufficient condition, since* (26) *is compatible with the divergence of the series* (23).

The parenthetical remark following (24) implies that (iv) admits functions $f(x)$ which are not of bounded variation.

The parenthetical remark following (26) is contained in Lusin's theorem concerning trigonometric series which are absolutely convergent on a set of positive measure.

What concerns the sufficiency of (24) and (25), it is seen from the obvious inequality $2^{\nu(n)} \leqq d(n)$ and from (8), that (24) implies (22). On the other hand, according to Hardy, the assumption (25) is sufficient for the convergence of the series

$$\sum_{n=1}^{\infty} n^{1+\epsilon}(a_n^2 + b_n^2), \tag{27}$$

which, in view of Schwarz's inequality, is sufficient for (24) (it is understood that the positive ϵ occurring in (24), (25), (27) is not one and the same ϵ).

Accordingly, only the sufficiency of (22), and the insufficiency of (26), for (23) need to be considered. The insufficiency of (26) will be proved by showing that (26) is compatible with the divergence of the simplest case of (23), namely, with the divergence of the numerical series which results by choosing $x = 0$ and $n = 1$ in (23). In view of (1), this numerical series is exactly the simplest case, $n = 1$, of Bruns' series (14). The proof of the sufficiency of (22) will merely require an application of Möbius' inversion, which, however, will now be based on (iii) and must therefore proceed in the direction which is precisely the opposite of the one occurring in the proof of (ii).

Even if merely (26) is assumed, it follows from (iii) that (17) is identical with $f_n(x)$ by virtue of (21) and (1). This means that

$$\beta_n = \sum_{k=1}^{\infty} \alpha_{nk}, \qquad (n = 1, 2, \cdots), \tag{28}$$

if, for any fixed x, the n-th term of the series (21) and the value of the function $f_n(x)$ are denoted by α_n and β_n respectively. Then (23) appears in the form

$$\alpha_n = \sum_{k=1}^{\infty} \mu(k)\beta_{nk}. \tag{29}$$

Finally, corresponding to the parenthetical remark made after (26), the assumptions (26), (22) mean the absolute convergence of the respective series $\sum \alpha_n$,

$\sum 2^{\nu(n)}\alpha_n$. Accordingly, the sufficiency of (22), and the insufficiency of (26), for (23) are equivalent to the following assertions:

(v) *Let* $\alpha_1, \alpha_2, \cdots$ *be a sequence of numbers satisfying*

$$\sum_{n=1}^{\infty} |\alpha_n| < \infty, \tag{30}$$

and let the sequence $\beta_1, \beta_2, \cdots$ *be defined by the series*

$$\beta_n = \sum_{n|k} \alpha_k, \qquad (n = 1, 2, \cdots). \tag{31}$$

Then Möbius' inversion

$$\alpha_n = \sum_{n|k} \mu(k/n)\beta_k, \qquad (n = 1, 2, \cdots), \tag{32}$$

(I) *is legitimate if*

$$\sum_{n=1}^{\infty} 2^{\nu(n)} |\alpha_n| < \infty \tag{33}$$

holds for the number, $\nu(n)$, *of the distinct prime divisors of* n *(and so, in particular, if*

$$\sum_{n=1}^{\infty} n^{\epsilon} |\alpha_n| < \infty \tag{34}$$

holds for some $\epsilon > 0$);

(II) *is not in general legitimate, since* (30) *does not imply the convergence of all the series* (32) *or, for that matter, of the series*

$$\sum_{k=1}^{\infty} \mu(k)\beta_k, \tag{35}$$

which belongs to $n = 1$.

10. The proof of (I) depends merely on Weierstrass' double-series theorem. And what Bruns (who originally was a *doctorandus* of Weierstrass) must have had in mind when claiming the sufficiency of (30), was undoubtedly an application of this theorem. However, what actually happens is that a summation of (32) with respect to n accumulates various divisors much before Weierstrass' theorem becomes applicable.

In fact, if n in (31) is replaced by nm and if the result is multiplied by $\mu(m)$, all that follows is that

$$\sum_{m=1}^{\infty} \mu(m)\beta_{nm} = \sum_{m=1}^{\infty} \mu(m) \sum_{k=1}^{\infty} \alpha_{nmk} \tag{36}$$

holds, provided that the series on the left of this relation is convergent (and it will be seen from the proof of (II) below that this proviso is not implied by (30) alone). In addition, the double-series theorem becomes applicable on the right of (36) only if

$$\sum_{q} \sum_{k=1}^{\infty} |\alpha_{nqk}| < \infty,$$

where q runs through all square-free integers and n is arbitrary.

Since $\nu(m)$ denotes the number of the distinct prime divisors of m, the number of all square-free divisors of m clearly is $2^{\nu(m)}$. Hence, a trivial counting contracts the last double series into the simple series

$$\sum_{m=1}^{\infty} 2^{\nu(m)} \mid \alpha_{nm} \mid ,$$

where n is arbitrarily fixed. This series is identical with (33) if $n = 1$, and it is majorized by

$$\sum_{m=1}^{\infty} 2^{\nu(nm)} \mid \alpha_{nm} \mid$$

for every n, since $\nu(m) \leqq \nu(nm)$. Since the last series is a subseries of (33), it follows that the applicability of Weierstrass' theorem to (36) requires precisely (33).

If (33) is satisfied, then the series on the left of (36) converges (in fact, absolutely) and, in addition, its representation by the double sum (36) can be rearranged into

$$\sum_{j=1}^{\infty} \alpha_j \sum_{k \mid j/n} \mu(k),$$

where the interior summation is vacuous or extends over all divisors k of the quotient j/n, according as the latter is not or is an integer. In either case, it is seen from (13) that the interior sum multiplying α_j is 1 or 0 according as j is or is not equal to n. Since the last formula line represents the value of the series on the left of (36), it follows that the sum of the series is α_n. In other words, (36) can be written in the form (32). This proves (I).

11. In order to prove (II), it is sufficient to show that (30) does not imply the convergence of the series (35), in which $\beta_1, \beta_2, \cdots$ are defined by the sums (31); sums the convergence of which is ensured by the assumption (30). The proof of this negation will be based on an adaptation of Toeplitz's norm-principle concerning linear summation processes of sequences.

If s_n denotes the n-th partial sum of the series (35), then

$$s_n = \sum_{m=1}^{n} \mu(m)\beta_m = \sum_{m=1}^{n} \mu(m) \sum_{m \mid k} \alpha_k ,$$

by (31). In view of (30), this double sum is convergent and can be rearranged arbitrarily. In particular, it can be arranged corresponding to distinct elements of the sequence $\alpha_1, \alpha_2, \cdots$. This arrangement gives

(37) $$s_n = \sum_{k=1}^{\infty} \phi_k(n)\alpha_k ,$$

if $\phi_k(n)$ is an abbreviation for the finite sum

(38) $$\phi_k(n) = \sum_{d \mid k}^{d \leqq n} \mu(d)$$

(in which the summation index runs through those divisors d of k which do not exceed n).

The assertion to be proved, namely, that the convergence of (35) is not implied by (30), means that there exist sequences $\alpha_1, \alpha_2, \cdots$ which satisfy (30) but are such that the corresponding sequence $s_1, s_2, \cdots$, defined by the linear transformation (37), fails to have a (finite) limit, $\lim s_n$. But it is clear from the general norm-principle of linear transformations, that the existence of such sequences $\alpha_1, \alpha_2, \cdots$ will be proved if it is shown that the least upper bound of the sequence $|\phi_1(n)|, |\phi_2(n)|, \cdots$ is not a bounded function of n. Accordingly, (II) will be proved if it is shown that there does not exist a constant C satisfying

$$|\phi_k(n)| < C, \tag{39}$$

where k and n are arbitrary. But (39) can be disproved by a slight modification (or, rather, by a reversal) of an elementary argument recently applied to a related question ([16], §13).

First, if p_m denotes the m-th prime, then p_m is of the order of $m \log m$. Even if this is meant just in Chebyshev's sense, it follows that, for every fixed h, the square of p_m exceeds p_{m+h} when m is large. This implies that, corresponding to any given positive integer h, there exist h distinct primes, say $r_1, \cdots, r_h$, which have the property that the greatest of them is less than the square of the least of them. (The more precise information supplied by the above conclusion, namely, that the primes $r_1, \cdots, r_h$ can be chosen as consecutive primes, will not be needed.)

Let k denote the product of the h primes $r_1, \cdots, r_h$, where $r_1 < \cdots < r_h$. Then, since $r_1^2 > r_h$, every composite divisor of k is greater than any of the prime divisors of k. Hence, from (38),

$$\phi_k(r_h) = \mu(1) + \sum_{j=1}^{h} \mu(r_j), \tag{40}$$

and so, since every r_j is a prime, $\phi_k(r_h) = 1 - h$. This refutes (39), since h can be chosen arbitrarily large.

This completes the proof of (v), and therefore that of (iv).

Remark. If (v) is particularized to the case of completely multiplicative coefficients, considered in the italicized statement (II) preceding (iv), then, since $f(x)$ satisfies the "Bernoullian" functional equation, $f_{nk}(x)$ in (23) becomes $f(nkx)$ times a multiplicative constant. Thus (23) becomes a series of the particular type for which Hartman and I developed a more or less comprehensive treatment (which, however, does not take into account expansions of the type of Kluyver's series, involving the Prime Number Theorem and its refinements); cf. [6], where further references are given. It is interesting from the historical point of view that this development was initiated (as late as 1937) by the example of an arithmetical trigonometric series mentioned by Riemann [13] in order to illustrate the

limitations of his own Fourier theory, involving R-integrals; and that his arithmetical example and its generalizations proved to be easily tractable by, and of course substantially depending on, Lebesgue's theory of Fourier series, based on L-integrals.

12. In view of the comments made in connection with (18), all the above considerations may be said to depend on Poisson's Fourier approach to the Euler-Maclaurin sum-formula. A related scheme, which again happens to be more familiar to astronomers than to mathematicians, depends on a connection between the equidistant trigometric interpolation, rather than the equidistant Riemannian approximations (1), of $f(x)$ and the Fourier series of $f(x)$, where $f(x) = f(x + 1)$. This connection is quite different from, and actually corresponds to the "Möbius inversion" of, the connection on which Euler's and Lagrange's interpolatory approach to Fourier series depends.

The latter connection may be expressed as follows: If n^* denotes $2n + 1$ for a fixed n, then the n^*-ary cyclic matrix of the n^*-th roots of unity $e(-kj/n^*)$, where k and j range from $-n$ to n, (mod n^*), and $e(t)$ denotes $e^{2\pi i t}$, is a unitary matrix except for a numerical factor, $(n^*)^{-\frac{1}{2}}$. And the linear transformation determined by the matrix maps the vector of the n^* equidistant values $f(m/n^*)$ of the function $f(x) = f(x + 1)$, where $m = 1, 2, \cdots, n^*$, on the vector of the n^* values which result if each of the Fourier constants

$$c_k = \int_0^1 f(x)e(-kx)\,dx \tag{41}$$

of $f(x)$ is replaced by its n^*-th equidistant Riemannian approximation, that is, by

$$c_k(n) = \sum_{m=1}^{n^*} f(m/n^*)e(-mk/n^*)/n^*; \qquad k = -n, \cdots, n. \tag{42}$$

Clearly,

$$f^n(x) = \sum_{k=-n}^{n} c_k(n)e(kx) \tag{43}$$

is that trigonometric polynomial of degree n (that is to say, of a degree not exceeding n) which interpolates the given function $f(x)$ at the n^* equidistant points $x = k/n^*$:

$$f^n(m/n^*) = f(m/n^*); \qquad m = 1, \cdots, n^*. \tag{44}$$

Since (44) suggests that (in some sense)

$$f^n(x) \to f(x) \quad \text{as } n \to \infty \tag{45}$$

(if the function $f(x)$ is smooth enough), and since (42) and (41) imply that

$$c_k(n) \to c_k \quad \text{as } n \to \infty \tag{46}$$

(for every *fixed* k), Euler was thus led from the elementary trigonometric identity (43) to the existence of a Fourier expansion

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k\, e(kx) \tag{47}$$

(under reservations to be specified). Curiously enough, the writings of Lagrange do not appear to contain this approach to Fourier series, although he had (42), (43), (44) explicitly and also was one of the principal participants in the celebrated discussions (centering about the partial differential equation $u_{tt} = u_{xx}$) as to the existence of a Fourier expansion for an "arbitrary" function.

Corresponding to the inversion (23) of (1), one of the issues arising in the numerical practice (cf. [1], §83 – §86) is not the above transition, (45), from (43) to (47), but the inverse problem: Given the Fourier constants (41) of the function (47), calculate the Fourier constants (42) of the interpolating functions (43); thus replacing the *limiting* connection (46), which is not under numerical control, by explicit relations, which apply to any *fixed* (in the numerical practice, large) value of n.

The answer to this question is known (*ibid.*, where only some of the formulae, taking into account the necessary computational preferences, but not the actual rules, are cumbersome). It may be formulated as follows:

(iii bis) *If a trigonometric series*

$$\sum_{-\infty}^{\infty} c_k\, e(kx), \qquad \left(\sum_{-\infty}^{\infty} = \lim_{N\to\infty} \sum_{-N}^{N}\right), \tag{47}$$

which need not be a Fourier series, converges for all x, *and if* $f(x) = f(x + 1)$ *denotes its sum, then the coefficients*, (42), *of the trigonometric polynomial* (43) *defined by* (44) *can be calculated from the* $n^* = 2n + 1$ *series*

$$c_k(n) = \sum_{j=-\infty}^{\infty} c_{k+jn^*}\,, \qquad k = 0, \pm 1, \cdots, \pm n \tag{48}$$

(*the convergence of which is part of the statement*).

This rule parallels (iii). Correspondingly, its verification is similar to the proof of (iii). In fact, since (47) is the definition of the function $f(x)$,

$$f(m/n^*) = \sum_{h=-\infty}^{\infty} c_h\, e(mh/n^*).$$

Hence, from (42),

$$c_k(n) = \sum_{h=-\infty}^{\infty} c_h \sum_{m=1}^{n^*} e(m(h - k)/n^*)/n^*.$$

But the interior sum on the right, being the mean-value of the $(h - k)$-th powers of all n^*-th roots of unity, is 1 or 0 according as $h - k$ is or is not a multiple of n^*, i.e., according as there does or does not exist a real integer j satisfying $h = k + jn^*$. Consequently, the last formula line is identical with (48).

13. The additive connection (48) between the harmonic analysis of $f(x)$ and that of the interpolating function $f^n(x)$ seems to go back to Euler and is sometimes used, for numerical purposes, not only in astronomy but in the less mathematically minded domain of meteorology also. On the other hand, this simple identity appears to have been neglected in pure mathematics, although it has implications of analytical interest. In this direction, the simplest of the conclusions which may be drawn from (48) is the following fact, which parallels the first assertion of (iv):

(iv bis) *If a continuous function* $f(x) = f(x + 1)$ *has a Fourier series*

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e(kx) \tag{49}$$

satisfying

$$\sum_{k=-\infty}^{\infty} | c_k | < \infty, \tag{50}$$

and if $f^n(x)$ *denotes the trigonometric polynomial, of degree* n, *which attains at the* $2n + 1$ *equidistant points* $x = m/(2n + 1)$, *where* $m = 0, \cdots, 2n$, *the same values as the function* $f(x)$, *then* $f^n(x) \to f(x)$ *as* $n \to \infty$, *uniformly in* x.

In view of the negative result of (iv), it is of interest that the mere absolute convergence of (49) is sufficient this time. This cannot be concluded by the usual method of obtaining criteria sufficient for (45), the reason being as follows:

When dealing with cases of uniform convergence in (45), the usual method assumes that, corresponding to every n, there exists *some* trigonometric polynomial, say $g^n(x)$, which is of degree n (at most) and which tends, when $n \to \infty$, to $f(x)$ so rapidly that the inequalities $| f(x) - g^n(x) | < \epsilon_n$ are satisfied by a sequence of constants $\epsilon_1, \epsilon_2, \cdots$, to be chosen so as to compensate for the unboundedness of the (uniform) "Lebesgue norms" of the process of equidistant trigonometric interpolation. But the n-th of these norms, though it is $O(\log n)$, is not $o(\log n)$; cf., e.g., [9], where further references are given. Hence, a sequence $\epsilon_1, \epsilon_2, \cdots$ can be admitted if and only if $\epsilon_n = o(1/\log n)$. Accordingly, the method is applicable to a function $f(x)$ if and only if some trigonometric polynomial $g^n(x)$ of degree n approximates $f(x)$ with an error which, uniformly in x, is $o(1/\log n)$ as $n \to \infty$. However, the impossibility of such an approximation is easily compatible with (50). In fact, *corresponding to any sequence of constants* ϵ_n *satisfying* $0 < \epsilon_n \to 0$, *there exists a continuous function* $f(x)$ *possessing an absolutely convergent Fourier series and having the property that, if* $| f(x) - g^n(x) | < \epsilon_n$ *holds for every* x *and for every* n, *then* $g^n(x)$ *cannot be a trigonometric polynomial of degree* n *for every* n. This is readily ascertained by considering such functions as $f(x) = \sum n^{-2} \cos m_n x$, where $m_1 < m_2 < \cdots$ is *any* lacunary sequence of integers.

Accordingly, the sufficiency of (50) for (45) is not contained in what is obtainable by the method of the "Lebesgue norms." Conversely, the latter method can apply when (50) is not satisfied. This follows from well-known results of

S. Bernstein, according to which the case $\epsilon = 0$ of (25) is insufficient for the absolute convergence of the Fourier series of $f(x)$, although the n-th "best approximation" to $f(x)$ is of a lower order than $1/\log n$ even if the $\frac{1}{2}$, rather than just the ϵ, is omitted from the exponent on the right of (25).

In order to prove the sufficiency of (50), let $s_n(x)$ denote the n-th partial sum of (49). Then, if (48) is inserted into (43), it is seen that

$$f^n(x) - s_n(x) = \sum_{k=-n}^{n} e(kx) \sum_{j=-\infty}^{\infty}{}' c_{k+jn^*},$$

where the prime signifies the omission of the summation index $j = 0$. Since $n^* = 2n + 1$, it follows that

$$|f^n(x) - s_n(x)| \leqq \sum_{k=-n}^{n} \sum_{m=1}^{\infty} |c_{k+(2n+1)m}| + \sum_{k=-n}^{n} \sum_{m=1}^{\infty} |c_{k-(2n+1)m}|.$$

Since k is restricted to the $2n + 1$ integers contained between $-n$ and n, and since m is not less than 1, it is clear that the subscripts of the Fourier constants c occurring in either of the last of the double sums represent *distinct* integers, and that all these integers are greater than n in the first, and less than $-n$ in the second, of the double sums. Accordingly, the last formula line implies that the difference $f^n(x) - s_n(x)$ is majorized by the n-th remainder term of (50). On the other hand, since $s_n(x)$ is the n-th partial sum of (49), the difference $f(x) - s_n(x)$ is majorized by the same remainder term of (50). Consequently,

$$\text{(45 bis)} \qquad |f^n(x) - f(x)| \leqq 2 \sum_{m=n+1}^{\infty} (|c_m| + |c_{-m}|).$$

This inequality proves (iv bis). In addition, it supplies an explicit estimate for the error term of (45).

I do not know whether (45) must, or need not, hold for every absolutely continuous $f(x) = f(x + 1)$. If it need not, the sufficiency of (50) is of particular interest, since (50) is satisfied by (49) whenever $f(x) = f(x + 1)$ is the indefinite integral of a function which, instead of being just of class (L), is of class (L^2).

14. The formal connection between the inversion (23) of (21) and (1), where $n = 0$ is excluded, and the inversion (48) of (47) and (43), where $k = 0$ is allowed, raises the question as to an analogue of (23) for the case $n = 0$ of the mean-value $\frac{1}{2}a_0 = c_0$. Since (23) then becomes what results from (14) after division by the factor n on the left, which is 0, it is indicated that the commensurabilities leading to the appearance of the denominator 0 must be eliminated by the introduction of incommensurabilities, before there is a possibility for the evaluation of $\frac{1}{2}a_0 = c_0$ as an infinite series involving Möbius' factors $\mu(m)$ as coefficients. And the commensurabilities responsible for the denominator $n = 0$ clearly are introduced by the fact that, in (1), the division of the unit interval is an equidistant Riemannian partition.

Thus it is natural to think of the Riemann sums

$$f_{kn}(0) = \sum_{m=1}^{kn} f(m/kn)/kn, \tag{51_1}$$

which are assigned by (1) and occur in (14), as degenerations of the Kronecker-Weyl (or, rather, Jacobi-Bohl) sums

$$f_{xn}(0) = \sum_{m=1}^{n} f(mx)/n, \tag{51_2}$$

where x is any fixed real number incommensurable with the period, 1, of $f(t)$. In fact, both (51_1) and (51_2) tend to the mean-value $\frac{1}{2}a_0 = c_0$ of $f(x) = f(x + 1)$ as $n \to \infty$ (whenever $f(x)$ is R-integrable). It turns out that the replacement of (51_1) by (51_2) actually leads to a representation of the mean-value (2) in terms corresponding to the case $n = 0$ of the evaluation (14).

THEOREM. *If $f(x)$ is an R-integrable function of period* 1, *then, in terms of Möbius' coefficients $\mu(m)$,*

$$\int_0^1 f(t)\, dt = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{d \mid n} \mu\left(\frac{n}{d}\right) f(xd) \tag{52}$$

holds for every irrational x.

The emphasis is not on the formal correctness of the identity (52), but on the convergence of the series (52), which lies quite deep. In fact, its proof will involve the Prime Number Theorem. And this cannot be avoided, since the assertion to be proved contains the Prime Number Theorem.

In order to see this, it is sufficient to select any irrational number x on the interval $0 < x < 1$, and to define an R-integrable function $f(t)$, of period 1, by placing $f(t) = 1$ when $t = x$, and $f(t) = 0$ when either $0 < t < x$ or $x < t < 1$ (the value of $f(0) = f(1)$ is always immaterial, since it does not occur at all in the series). For this function, the interior sum on the right of (52), where the summation index d runs through all divisors of n, becomes just $\mu(n/1)f(x \cdot 1) = \mu(n)$; so that (52) appears in the form $0 = \sum \mu(n)/n$. But the convergence of the series $\sum \mu(n)/n$ is equivalent to the Prime Number Theorem.

For any sequence of numbers $g(1), g(2), \cdots$, consider, on the one hand, its average

$$M(g) = \lim_{n\to\infty} \sum_{m=1}^{n} g(m)/n \tag{53}$$

(provided that (53) exists as a finite limit) and, on the other hand, the series

$$\sum_{n=1}^{\infty} g'(n)/n \tag{54}$$

(possibly divergent), in which $g'(1), g'(2), \cdots$ denotes the sequence of numbers defined by

$$g'(n) = \sum_{d \mid n} \mu(n/d) g(d). \tag{55}$$

It was recently shown ([16], (vii) on p. 6 and (2) on p. 1, finally (i) on p. 3) that, if $g(1)$, $g(2)$, $\cdots$ is any bounded sequence (or, more generally, any sequence bounded from below), then the existence of a finite limit (53) implies, and is implied by, the convergence of the series (54); and that the sum of the series (54) is always the value of the average (53) of $g(n)$.

Let this be applied to the sequence $g(n) = f(nx)$. Then, since x is irrational and $f(t)$ is R-integrable, the Kronecker-Weyl approximation (51_2) tends to the integral (52), as $n \to \infty$. This means that the average (53) exists and is equal to the integral (52). On the other hand, (55) shows that (54) becomes the series (52). Hence, it is sufficient to ascertain that $g(n) = f(nx)$ is a bounded function of n. But this is obvious, since every R-integrable function $f(t) = f(t+1)$ is bounded.

15. It follows that *every Fourier constant* (41) *of any R-integrable function* $f(t)$ *of period* 1 *can be represented by the convergent series*

$$c_k = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{d \mid n} \mu(d) f\left(\frac{nx}{d}\right) e^{-2\pi i k n x/d}, \tag{56}$$

where x is an arbitrary irrational number (of which c_0, $c_{\pm 1}$, $c_{\pm 2}$, $\cdots$ are, therefore, independent; the values of $f(t)$ attained at rational t do not occur in these series). In the case of the mean-value c_0, this evaluation becomes exactly (52). Actually, the evaluation (56) of an arbitrary c_k contains nothing new, since it results if $f(t)$ in (52) is replaced by the integrand of (41).

More revealing than this generalization is the particularization of $f(t)$ to the first Bernoullian function,

$$t - [t] - \tfrac{1}{2} = -\pi^{-1} \sum_{m=1}^{\infty} \sin 2\pi m t/m, \qquad (t \neq 0, \pm 1, \cdots). \tag{57}$$

In fact, it may be expected that, just as in the elementary theory of Fourier series, the case of this single function contains the substance of (52). But what thus results is the following

Corollary. *If $\phi^x(n)$ denotes the number of those elements of the n-th Farey group which do not exceed x (so that ϕ^1 is Euler's ϕ), then, for every irrational x satisfying $0 < x < 1$, the series*

$$\sum_{n=1}^{\infty} (x\phi(n)/n - \phi^x(n)/n) \tag{58}$$

is convergent and its sum, $\tfrac{1}{2}$, is independent of x.

This is somewhat unexpected from the methodical point of view. For example, an investigation of Davenport [2] might suggest that the convergence of such a series as (58) for *all irrational* x must depend on a *uniform* estimate, of the Vinogradow type, for the remainder terms of the Prime Number Theorems of *all* arithmetical progressions (each of which corresponds to a finite group of *rational* values of x). But it turns out that this heavy Diophantine machinery

is in no sense involved. In fact, in the proof of the sufficiency of the Tauberian restriction $|g(n)| <$ const., leading from the existence of the average (53) to the convergence of the series (54), only the existence of an $\epsilon > 0$ satisfying

$$\sum_{m=1}^{n} \mu(m) = o(n/\log^{1+\epsilon} n) \tag{59}$$

was used ([16], p. 6). But this estimate is just a slight refinement of the Prime Number Theorem ($\epsilon = -1$) of the full sequence of all integers; so that not even a finite group of L-series, except Riemann's zeta-function itself, is needed.

The function $\phi^x(n)$ defined before (58) is known to be identical with the finite sum

$$\phi^x(n) = \sum_{d \mid n} \mu(n/d)[xd] \tag{60}$$

(in fact, it is precisely this identity, the result of a straightforward counting, that underlies the connection established by Franel [4] between Farey's sections and Riemann's hypothesis). If $x = 1$, then (60) becomes Möbius' inversion of the definition of Euler's $\phi(n)$. Hence, if (60) is subtracted from $x\phi(n)$ for a fixed (irrational) x, it follows that (58) is precisely the case $f(t) = t - [t]$ of the series (52). Since the integral (52) becomes $\frac{1}{2}$ in this case, the proof is complete.

16. As explained before (51_1), the relation (52), which proved to be correct, is precisely what is suggested by the parallelism between $f_n(x)$ and $f^n(x)$. The same parallelism also suggests that, corresponding to the relationships between the degree of smoothness of a (very smooth) continuous function $f(x)$ and the order of the error term of (45), there result similar relationships if $f^n(x)$ is replaced by $f_n(x)$. This too proves to be correct.

In this direction, it will suffice to prove the following particular criterion (refinements of which will be clear enough from its proof):

(viii) *A function $f(x)$ of period* 1, *where x is real, is regular-analytic along the x-axis if and only if, for some positive $\theta =$* const. < 1, *the equidistant Riemannian sums* (1) *approximate the mean-value* (2) *with an error $o(\theta^n)$, uniformly in x, as $n \to \infty$.*

If this o-condition is weakened to $o(1/n^{1/\epsilon})$, where $\epsilon > 0$ is fixed but arbitrarily small, the resulting criterion is necessary and sufficient in order that all derivatives $f'(x), f''(x), \cdots$ of $f(x)$ exist for every x.

It is understood that the regular-analyticity of $f(x)$ is meant in the following sense: Corresponding to every x_0, there exists a positive $r = r(x_0)$ such that $f(x)$ is a power series in $x - x_0$, converging to $f(x)$ when $x_0 - r < x < x_0 + r$. Since $f(x) = f(x + 1)$, this will be the case if and only if there exist in the complex z-plane an open ring-shaped domain containing the circle $|z| = 1$ and a Laurent series in such a way that the latter converges within the ring and becomes identical with $f(x)$ when $z = e^{2\pi ix}$. And it is clear that this will be the case if and only if the convergence of the Fourier series (49) is as strong as that of some geometric progression (uniformly in x), i.e., if and only if there exists a positive $\theta =$ const.

< 1 for which both c_k and c_{-k} are $o(\theta^k)$ as $k \to \infty$. In the notations of (iv), this means that

$$a_k = o(\theta^k), \qquad b_k = o(\theta^k) \tag{61}$$

as $k \to \infty$. Hence, what the first of the two criteria (viii) asserts is that (61) is equivalent to

$$f_n(x) = \int_0^1 f(t)\,dt + o(\theta^n), \qquad \text{uniformly in } x, \tag{62}$$

as $n \to \infty$. It can, of course, be assumed that the value of the constant (2) is 0.

Under this assumption, suppose first that (61) is satisfied. Then, from (17) in (iii),

$$|f_n(x)| \leqq \sum_{k=1}^{\infty} (|a_{nk}| + |b_{nk}|) = \sum_{k=1}^{\infty} o(\theta^{nk}) = o(\theta^n).$$

This means that (62) is satisfied.

Conversely, if (62) is satisfied, then, since the integral on the right of (62) is supposed to vanish, (23) in (iv) shows that the n-th term of the series (21) is majorized by

$$\sum_{k=1}^{\infty} |\mu(k)f_{nk}(x)| \leqq \sum_{k=1}^{\infty} |f_{nk}(x)| = \sum_{k=1}^{\infty} o(\theta^{nk}) = o(\theta^n),$$

uniformly in x. But this means that (61) is satisfied.

The transcription of the proof to the case of the second of the criteria (viii) requires only trivial alterations, since a periodic function has derivatives of arbitrarily high order if and only if its Fourier constants are

$$a_n = o(1/n^{1/\epsilon}), \qquad b_n = o(1/n^{1/\epsilon}) \tag{61 bis}$$

for every fixed $\epsilon > 0$. For further criteria of the type (61), (61 bis), cf., e.g., [15].

17. It is clear from the definition of $\phi^x(n)$ in the Corollary of the Theorem, that the convergence of the series (58), to a sum which is independent of the irratonal x, is an elaborate manifestation of the asymptotic equidistribution of Farey's sequence. The asymptotic equidistribution itself is less deep; all that it states is that, if r runs through the $\phi(n)$ positive integers relatively prime to, and not greater than, a fixed n, then

$$\sum_r f(r/n)/\phi(n) \to \int_0^1 f(t)\,dt \tag{I}$$

as $n \to \infty$ holds for every R-integrable function $f(t)$. And (I) is an elementary lemma of Pólya ([10], p. 75).

(I) is worth proving in a manner corresponding to the method used above. First, the truth of (I) for every R-integrable $f(t)$, being equivalent to an asymp-

totic equidistribution in the sense of Weyl, is equivalent to the truth of (I) for each of the particular functions $f(t) = e(kx)$, where $k = 0, \pm 1, \cdots$. In other words, (I) is equivalent to the sequence of limit relations $c_n(1)/\phi(n) \to 0$, $c_n(2)/\phi(n) \to 0, \cdots$, where $c_n(m)$ denotes the sum of the m-th powers of all the n-th primitive roots of unity, the so-called Ramanujan sum. But $c_n(m) = O(1)$ as $n \to \infty$ holds for every fixed m (cf. [12]). Hence, in order to complete the proof, it is sufficient to observe that $\phi(n) \to \infty$ as $n \to \infty$.

Actually, (I) can be refined in terms of a Fourier analysis, if the considerations based on the formal fact (iii) are transferred from the equidistant Riemannian sums (1) to the corresponding sums

$$\text{(II)} \qquad {}_nf(x) = \sum_r f(x + r/n)/\phi(n),$$

where $f(x)$ is of period 1 (the integer r runs through its $\phi(n)$ values). To this end, it will be convenient to use Ramanujan's notation $c_n(m)$ for all real, rather than just for positive, values of the integer m. This obviously gives $c_n(m) = c_n(-m)$ if $m < 0$ and $c_n(m) = \phi(n)$ if $m = 0$. It is also clear that every $c_n(m)$ is an integer.

In order to avoid a confusion of Ramanujan's c's with the c's in (47), let the complex Fourier constants of an L-integrable function $f(x) = f(x + 1)$ be now denoted by α's:

$$\text{(III)} \qquad f(x) \sim \sum_{k=-\infty}^{\infty} \alpha_k e(kx).$$

If x is replaced by $x + r/n$ in (III), it is seen from the definition of $c_n(m)$ that the Fourier series of the function (II) of x is

$$\text{(IV)} \qquad {}_nf(x) \sim \sum_{k=-\infty}^{\infty} c_n(k)\alpha_k e(kx)/\phi(n).$$

This corresponds to (iii).

If $f(x)$ is R-integrable, then (I) shows that (II) tends to the limit (2) as $n \to \infty$; so that, if $\frac{1}{2}a_0$ denotes the mean-value, α_0, of (III),

$$\text{(V)} \qquad {}_nf(x) \to \tfrac{1}{2}a_0 \qquad (\tfrac{1}{2}a_0 = \alpha_0)$$

as $n \to \infty$ holds for every x. On the other hand, if the Fourier series (III) converges to $f(x)$ at every x (for instance, if $2f(x) = f(x + 0) + f(x - 0)$ holds at every x and f is of bounded variation), then the sign of equivalence can be replaced by the sign of equality in (IV) also. Then, if α_k in (III) is replaced by the "real" Fourier constants a_k, b_k, it follows from $\alpha_0 = \frac{1}{2}a_0$ that the equivalence (IV) becomes the identity

$$\text{(VI)} \qquad {}_nf(x) - \tfrac{1}{2}a_0 = \sum_{k=1}^{\infty} c_n(k)(a_k \cos 2\pi kx + b_k \sin 2\pi kx)/\phi(n).$$

This implies the following curiosity relating to the convergence problem of Fourier series:

If a trigonometric series

$$\tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \cos kx)$$

is convergent at every x and if its sum, $f(x) = f(x + 2\pi)$, is R-integrable, then the series

$$\sum_{k=1}^{\infty} c_n(k)(a_k \cos kx + b_k \sin kx)$$

converge to values which are $o(\phi(n))$ uniformly in x.

This follows from (VI), since, as easily seen from (2) and from the proof of (I), the error term of (V) is $o(1)$ uniformly in x.

The analogue of (58) results by choosing $a_k = 0$ and $b_k = -1/\pi k$. Then (III) becomes (57). Hence, (II) shows that (VI) appears in the form

(58 bis) $$\sum_r (r/n + x) - \sum_r [r/n + x] - \tfrac{1}{2}\phi(n) = -1/\pi \sum_{k=1}^{\infty} c_n(k)/k \sin 2\pi kx,$$

where $x \neq 0, \pm 1, \cdots$; so that what follows is

(58*) $$\sum_{k=1}^{\infty} c_n(k)/k \sin 2\pi kx = o(\phi(n)), \qquad n \to \infty,$$

uniformly in x. The uniformity is noteworthy in view of the Gibbs phenomenon of (57) itself.

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